ON THE ORDER OF VANISHING OF THE CYCLOTOMIC p-ADIC L-FUNCTION

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Abstract. For a newform f for $\Gamma_0(N)$ of even weight k, we prove that the p-adic L-function $L_p(f,\alpha)$ is not identically zero on the group $\mathbb{Z}_p^*$ of the p-adic units. If $p \geq 5$, we prove that the order of vanishing of $L_p(f,\alpha)$ at any p-adic integer is finite.

Introduction

To any eigenform $f \in S_k(\Gamma_0(N))$ of the Hecke operator T_p, Mazur-Tate-Teitelbaum [6] attached a p-adic L-function $L_p(f,\alpha;\chi)$ defined over $\mathcal{X} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p)$, the group of p-adic continuous characters. As a topological group, $\mathcal{X}$ is isomorphic to $\mu_{p-1} \times \mathbb{Z}_p$, where μ_{p-1} stands for the group of $(p-1)$-th roots of unity. From a theorem of Rohrlich ([8, Theorem 1]), it follows that if f is a normalized newform, $L_p(f,\alpha;\chi)$ does not vanish identically over $\mathcal{X}$, a conjecture stated for $k=2$ by Mazur and Swinnerton-Dyer ([5, Conjecture 1]).

The present article deals with the restriction of $L_p(f,\alpha;\chi)$ to the characters $\chi_s(x) = \langle x \rangle^s$, $s \in \mathbb{Z}_p^*$. We prove that the function $L_p(f,\alpha)(s) = L_p(f,\alpha;\chi_s)$ has a non-zero derivative at any p-adic integer s when f is a normalized newform. In particular, we obtain that $L_p(f,\alpha)$ does not vanish identically over $\{1\} \times \mathbb{Z}_p$, a refinement of Mazur and Swinnerton-Dyer's conjecture suggested by Henri Darmon.

In section 1, we review the construction of $L_p(f,\alpha;\chi)$, following [6]. In section 2, we apply an ultrametric analogue of the Stone-Weierstrass theorem to establish that $L_p(f,\alpha)$ does not vanish identically over $\mathbb{Z}_p$, provided that $p \geq 3$. If f has weight 2, we can say that, actually, $L_p(f,\alpha)$ does not vanish identically over $\mathbb{Z}_p^*$. In the ordinary case, the finiteness of the set of

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zeros of $L_p(f,\alpha)$ also follows, as well as the fact that if p is a primitive root mod N, and $\alpha \not\equiv 1 \pmod{p}$, this function does not vanish identically mod p, provided that not all the modular integrals are zero mod p.

In section 3, we use the above non-vanishing results in conjunction with some estimations of the moments of $L_p(f,\alpha)$ to prove our main result, namely, that the order of vanishing of $L_p(f,\alpha)(s)$ at any p-adic integer s is finite, if $p \geq 5$ (Theorem 3.1). In particular, it is finite at the central point $s = k/2$.

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Contents

1. The cyclotomic p-adic L-functions

1.1. **Modular integrals of cusp forms.** We fix p a prime number and embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. By $\mathcal{O}_p$, we denote the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}_p$. We consider the p-adic norm $|\ |$ on $\mathbb{C}_p$ normalized so that $|p| = p^{-1}$.

The set of elements in $\mathbb{Z}_p^*$ congruent to an integer a mod p^n, $p \nmid a$, will be denoted by $D(a, p^n)$.

Let $S_k(\Gamma_0(N))$ be the $\mathbb{C}$-vector space of cusp forms of positive even weight k for $\Gamma_0(N)$. Let $\mathbb{T}$ be the $\mathbb{Z}$-algebra spanned by the Hecke operators $\{T_n\}_{n\geq 1}$ acting on it. In much of the paper, we will suppose that $f = \sum_{n\geq 1} a_n(f)q^n$ is a normalized newform. In this case, the number field $K_f = \mathbb{Q}(\{a_n(f)\})$ is totally real. Its ring of integers will be denoted by $\mathcal{O}_f$.

For the complex L-function $L(f, s)$, the following identity holds:

$$\Lambda(f, s) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(f, s) = N^{\frac{s}{2}} \int_0^\infty f(it)t^{s-1}dt, \tag{1.1}$$

for $s \in \mathbb{C}, \mathrm{Re}(s) > \frac{k}{2} + 1$. The right hand side of the equality defines an entire function which satisfies the following functional equation:

$$\Lambda(f, s) = \pm\Lambda(f, k - s), \quad s \in \mathbb{C}. \tag{1.2}$$

If χ is a primitive Dirichlet character of conductor n, $(n, N) = 1$, and $\tau(\chi)$ its attached Gauss sum, the complex L-function of f twisted by χ is defined by

$$L(f, \chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)a_n(f)}{n^s}, \quad \mathrm{Re}(s) > \frac{k}{2} + 1.$$

Let $\Lambda(f, \chi, s) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(f, \chi, s)$. By using orthogonality properties of the Dirichlet characters, we have that

$$\Lambda(f, \chi, s) = N^{\frac{s}{2}} \frac{\tau(\chi)}{n} \sum_{a=0}^{n-1} \overline{\chi(a)} \int_0^\infty f\left(\frac{a}{n} + it\right) t^{s-1}dt, \quad \mathrm{Re}(s) > \frac{k}{2} + 1. \tag{1.3}$$

The function $\Lambda(f, \chi, s)$ extends to an entire function and satisfies a functional equation similar to (1.2).

For $r \in \mathbb{Q}$ and $0 \leq j \leq k - 2$, the integrals

$$\lambda(f, r, j) = 2\pi \int_0^\infty f(r + it)t^j dt, \quad r \in \mathbb{Q},$$

are known in the literature as modular integrals. They satisfy the following

Theorem 1.1. ([6, § 2]) *There exists a $\mathbb{Z}$-lattice Σ_f of finite rank such that $\lambda(f, r, j) \in \Sigma_f$.*

It is an interesting question to study when the twisted L-function of a modular form vanishes. The following theorem is a result in this direction.

Theorem 1.2. (Rohrlich [8, Theorem 1]) *Let $f \in S_k(\Gamma_0(N))$ be a normalized newform. Let P be a finite set of primes and X_P the set of primitive Dirichlet characters unramified outside $P \cup \{\infty\}$. Then, for all but finitely many $\chi \in X_P$, $L\left(f, \chi, \frac{k}{2}\right) \neq 0$.*

Since any Dirichlet character χ of conductor p^n is unramified outside p, we deduce the following

Corollary 1.3. *For $n \in \mathbb{N}$ large enough there exists integers a_n, $p \nmid a_n$, such that*

$$\int_0^{\infty} f\left(\frac{a_n}{p^n} + it\right) t^{\frac{k}{2}-1} dt \neq 0\,.$$

1.2. **The p-adic L-function of a cusp form.** Let $f \in S_k(\Gamma_0(N))$ be an eigenform for T_p, $p \nmid N$, with eigenvalue $a_p(f)$. The Hecke polynomial attached to f at p is defined as follows:

$$X^2 - a_p(f)X + p^{k-1}.$$

For a root α of this polynomial and for $0 \leq j \leq k-2$, we may define an $\mathcal{O}_f[\frac{1}{\alpha}] \otimes \Sigma_f$-valued p-adic distribution by setting

$$\mu_{\alpha,j}(D(a,p^n)) = \frac{1}{\alpha^n}\left(\lambda\left(f, \frac{a}{p^n}, j\right) - \frac{1}{\alpha}\lambda\left(f, \frac{a}{p^{n-1}}, j\right)\right), \quad n \geq 1. \tag{1.4}$$

If $|a_p(f)| = 1$, it is said that f is ordinary at p. Otherwise, f is said to be supersingular at p. A root α is admissible (cf. [6]) if $p^{1-k} < |\alpha| \leq 1$. In the ordinary case, there is only one admissible root, α; it is a p-adic unit and $\mu_{\alpha,j}$ is then a p-adic measure. In the supersingular case, both roots are admissible, but the p-adic distributions are unbounded on the compact-open sets of $\mathbb{Z}_p^*$.

Definition 1.4. The cyclotomic distributions attached to f and p are defined as

- If f is ordinary at p: $\mu_{\alpha,j}$, where α is the unique root of the Hecke polynomial for f at p which is a p-adic unit.

- If f is supersingular at p: $\mu_{\alpha_i,j}$, $i = 1, 2$, where α_i denote the roots of the Hecke polynomial.

For an ordinary prime p, it is possible to integrate continuous $\mathbb{Q}_p$-valued functions by using uniform approximation by locally constant functions (cf. [4, Chapter 2]). The definition of an integral in the supersingular case requires some more work. First of all, we need to restrict the class of functions to be integrated.

Definition 1.5. A function $F : \mathbb{Z}_p^* \to \mathbb{Q}_p$ is said to be locally analytic if there is a covering of $\mathbb{Z}_p^*$ by compact-open sets $D(a, p^m)$ such that

$$F|_{D(a,p^m)}(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

The following theorem provides an integral operator attached to any admissible root.

Theorem 1.6. (Mazur-Tate-Teitelbaum, [6]) *Let $f \in S_k(\Gamma_0(N))$ be an eigenform of the Hecke operator T_p. For a compact-open subset $K \subseteq \mathbb{Z}_p^*$, there exists a unique $\overline{\mathbb{Q}}_p^r$-valued $\mathbb{Q}_p$-linear operator on the space of locally analytic functions, denoted by $\int_K F(x)d\mu_\alpha(x)$, with the following properties:*

(1) *Interpolation property: $\int_K x^j d\mu_\alpha(x) = \mu_{\alpha,j}(K)$, for $0 \le j \le k-2$.*
(2) *Divisibility property: for any $n \ge 0$*

$$\int_{D(a,p^n)} (x-a)^m d\mu_\alpha(x) \in \left(\frac{p^m}{\alpha}\right)^n \alpha^{-1}\Sigma_f \otimes \mathcal{O}_p.$$

(3) *Continuity property: if $F(x) = \sum_{n\ge0} c_n(x-a)^n$ in $D(a,p^m)$, then*

$$\int_{D(a,p^m)} F(x)d\mu_\alpha(x) = \sum_{n\ge0} c_n \int_{D(a,p^m)} (x-a)^n d\mu_\alpha(x).$$

(4) *For a fixed F, the assignment $K \mapsto \int_K F(x)d\mu_\alpha(x)$ yields a finitely additive function on the set of compact-open subsets of $\mathbb{Z}_p^*$.*

Let $\mathcal{X} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p)$. Since continuous characters are locally analytic, one can formulate the following

Definition 1.7. (The cyclotomic p-adic L-function) Assume that $p \nmid N$. Let $f \in S_k(\Gamma_0(N))$ be an eigenform of the Hecke operator T_p and α an admissible root of the Hecke polynomial of f at p. For $\chi \in \mathcal{X}$, we define

$$L_p(f, \alpha; \chi) = \int_{\mathbb{Z}_p^*} \chi(x) d\mu_\alpha(x).$$

1.3. **Special characters.** For $x \in \mathbb{Z}_p^*$, we write $x = \omega(x)\langle x \rangle$ where $\omega(x)$ is the unique $(p-1)$-th root of unity in $\mathbb{Z}_p^*$ congruent to x mod p and $\langle x \rangle \in D(1, p)$.

Let us now consider the characters of the form $\chi_s(x) = \langle x \rangle^s$, for $s \in \mathbb{Z}_p$. For an admissible root α, we denote

$$L_p(f, \alpha)(s) = L_p(f, \alpha; \chi_{s-1}) = \int_{\mathbb{Z}_p^*} \langle x \rangle^{s-1} d\mu_\alpha(x), \quad s \in \mathbb{Z}_p.$$

As in the complex analytic, a functional equation holds for $L_p(f, \alpha)$. In order to recall it, we first introduce some notation.

Let $K \subseteq \mathbb{Z}_p^*$ be a compact-open set and F a locally analytic function over $\mathbb{Z}_p^*$. Let us define the map $g(x) = -\frac{1}{Nx}$, $x \in \mathbb{Z}_p^*$. We set

$$\tilde{F}(x) = N^{\frac{k-2}{2}} x^{k-2} (F \circ g), \quad \tilde{K} = g(K), \quad \tilde{f} = w_N(f),$$

where w_N stands for the Fricke involution on $S_k(\Gamma_0(N))$.

Proposition 1.8. (cf. [6, Theorem 17.1]) *If α is an admissible root for f,*

$$\int_K F(x) d\mu_\alpha(x) = \int_{\tilde{K}} \tilde{F}(x) d\mu_\alpha(x).$$

Corollary 1.9. *Suppose that f is a newform. For any $s \in \mathbb{Z}_p$,*

$$L_p(f, \alpha)(s) = \pm \langle N \rangle^{1-s} \int_{\mathbb{Z}_p^*} x^{k-2} \langle x \rangle^{1-s} d\mu_\alpha(x).$$

In particular, if $k = 2$, $L_p(f, \alpha)(s) = 0$ if and only if $L_p(f, \alpha)(2 - s) = 0$.

Proof. Since f is a normalized newform, it is an eigenfunction for the Fricke involution; hence, $\tilde{f} = \pm f$, and we have

$$L_p(f, \alpha)(s) = \int_{\mathbb{Z}_p^*} \langle x \rangle^{s-1} d\mu_\alpha(x) = \pm \int_{\mathbb{Z}_p^*} x^{k-2} \left\langle \frac{-1}{Nx} \right\rangle^{s-1} d\mu_\alpha(x).$$

Since $\langle\ \rangle$ is a multiplicative group homomorphism, the result follows. $\square$

2. Non-vanishing results

If $p \nmid N$ and χ is a finite order p-adic character of conductor p^n, we have

$$L_p(\alpha, f; \chi\chi_j) = \frac{1}{\alpha^n}\left(1 - \frac{\overline{\chi(p)}p^{k-2}}{\alpha}\right)\left(1 - \frac{\chi(p)}{\alpha}\right)\frac{p^n}{\tau(\bar{\chi})}L(f, \bar{\chi}, j+1), \tag{2.1}$$

for $0 \leq j \leq k-2$ (cf. [6, pp. 20-21]). Rohrlich's theorem 1.2, together with identity 2.1, implies that $L_p(f, \alpha; \chi)$ does not vanish identically over $\mathcal{X}$. In this section we study the non-vanishing of $L_p(f, \alpha)$ over $\mathbb{Z}_p$.

2.1. **Approximation by polynomials.**

We denote by χ_G the characteristic function of a set $G \subseteq \mathbb{Q}_p$. From now on, K will denote a compact-open subset of $\mathbb{Q}_p$. The norm of a continuous function $F : K \to \mathbb{Q}_p$ is defined by

$$||F||_{\infty,K} = \max_{x \in K} |F(x)|\,.$$

For an integer $n \geq 1$, let $K[X^n]$ denote the set of polynomials with coefficients in K spanned by the monomials $\{X^{nm}\}_{m \geq 0}$. Let $\overline{K[X^n]}$ be its closure in $\mathcal{C}(K, \mathbb{Q}_p)$ with respect to the ∞-norm.

Definition 2.1. A function $F : K \to \mathbb{Q}_p$ is said to be a step function on K if there exists a finite family of compact-open subsets $G_i \subseteq K$, $1 \leq i \leq n$, such that $F = \sum_{i=1}^n a_i \chi_{G_i}$, with $a_i \in \mathbb{Q}_p$. A function $F : K \to \mathbb{Q}_p$ is said to be a locally polynomial function on K if for any $a \in K$ there exists a neighborhood G^a and a polynomial P_a such that $F|_{G^a}(s) = P_a(s)$, for any $s \in G^a$. If all the polynomials are of degree 0, we say that F is locally constant. We will denote by $\mathrm{Loc}(K, \mathbb{Q}_p)$ the set of locally constant functions on K.

Proposition 2.2. *A function $F : K \to \mathbb{Q}_p$ is a locally polynomial function on K if and only if there exists a finite covering of K by compact-open subsets $\{G_i\}_{1 \leq i \leq n}$ and polynomials $\{P_i\}_{1 \leq i \leq n}$ with coefficients in $\mathbb{Q}_p$ such that $F(x) = \sum_{i=1}^n P_i(x)\chi_{G_i}(x)$, for any $x \in K$. In particular, F is locally constant if and only if F is a step function.*

Theorem 2.3. ([1, Proposition 2.6]) *If A is a closed subalgebra of $\mathcal{C}^n(K, \mathbb{Q}_p)$ which separates points and contains the constant functions, then* $\mathrm{Loc}(K, \mathbb{Q}_p) \subseteq A$.

Corollary 2.4. *If A is a closed subalgebra of $\mathcal{C}^n(K,\mathbb{Q}_p)$ that separates points and contains the constant functions, then A contains the locally polynomial functions on K with coefficients in $\mathbb{Q}_p$.*

Proof. Let $F=\sum_{i=1}^n P_i\chi_{G_i}$ be a locally polynomial function with coefficients in $\mathbb{Q}_p$. For any $\varepsilon>0$, let us consider $\varepsilon_0=\varepsilon/\max\limits_{1\le i\le n}||P_i||_{\infty,K}$. By 2.3, there exist $g_i\in A$ such that $||g_i-\chi_{G_i}||_{\infty,K}<\varepsilon_0$. The result now follows. □

Corollary 2.5. $\mathrm{Loc}(\mathbb{Z}_p^*,\mathbb{Q}_p)\subseteq\overline{\mathbb{Z}_p^*[X^{p-1}]}$.

Proof. Since the function $F(x)=x^{p-1}$ is injective on $U=D(1,p)$, it follows from Theorem 2.4 that $\mathrm{Loc}(U,\mathbb{Q}_p)\subseteq\overline{U[X^{p-1}]}$, and since the balls $\{D(1,p^n)\}_{n\ge1}$ form a basis of the topology of U, for any $n\ge1$ we can find a sequence $(P_m(X))$ of polynomials in $U[X^{p-1}]$ such that

$$\lim_{m\to\infty}\max_{x\in D(1,p^n)}\left|P_m(x^{p-1})-\chi_{D(1,p^n)}(x)\right|=0.$$

But, given $1\le a\le p^n-1$ coprime to p, $\chi_{D(a,p^n)}(x)=\chi_{D(1,p^n)}(xa^{-1})$ and

$$\begin{aligned}\max_{x\in D(1,p^n)}&\left|P_m(x^{p-1})-\chi_{D(1,p^n)}(x)\right|\\ &=\max_{ax\in D(a,p^n)}\left|P_m(a^{1-p}(xa)^{p-1})-\chi_{D(1,p^n)}\left(\tfrac{ax}{a}\right)\right|\\ &=\max_{y\in D(a,p^n)}\left|Q_m(x^{p-1})-\chi_{D(a,p^n)}(y)\right|.\end{aligned}$$

Since $Q_m(X^{p-1})\in\mathbb{Z}_p^*[X^{p-1}]$ and the compact-open sets $\{D(a,p^n)\}$ form a covering of $\mathbb{Z}_p^*$, the result holds. □

2.2. **The non-vanishing on $\mathbb{Z}_p$.** We will use the following

Lemma 2.6. *Assume that $p>2$, let $f\in S_k(\Gamma_0(N))$ be a normalized newform and α an admissible root of the Hecke polynomial for f at p. If the p-adic integral operator attached to f, p and α vanishes over $\mathbb{Z}_p^*[X^{p-1}]$, then*

$$\int_{\mathbb{Z}_p^*}x^j\chi_{D(a,p^n)}(x)d\mu_\alpha(x)=0,$$

for any $0\le j\le k-2$, $n\ge1$, and any a coprime to p, $1\le a\le p^n-1$.

Proof. a) If f is ordinary at p, the unique admissible root α is a p-adic unit and $\mu_{\alpha,j}$ are p-adic measures. Thus, if we set $F_j(x) = x^j\chi_{D(a,p^n)}(x)$, for $x \in \mathbb{Z}_p^*$, we may choose a subset of polynomials $\{P_m\} \subseteq \mathbb{Z}_p^*[X^{p-1}]$ such that $\lim_{m\to\infty} ||P_m - F_j||_{\infty,\mathbb{Z}_p^*} = 0$. But $\int_{\mathbb{Z}_p^*} P_m(x)d\mu_\alpha(x) = 0$ for any $m \geq 1$, and since there exists $M > 0$ such that $|\mu_{\alpha,j}(K)| < M$ for any compact-open subset $K \subseteq \mathbb{Z}_p^*$, we have

$$\left|\int_{\mathbb{Z}_p^*} F_j(x)d\mu_\alpha(x)\right| = \left|\int_{\mathbb{Z}_p^*} (P_m(x) - F_j(x))\, d\mu_\alpha(x)\right| \leq M||P_m - F||_{\infty,\mathbb{Z}_p^*},$$

which tends to zero when $m \to \infty$.

b) If f is supersingular at p, none of the distributions is bounded and we need to change the strategy. We choose a subset of polynomials $\{P_m\} \subseteq \mathbb{Z}_p^*[X^{p-1}]$ such that

$$||P_m - F_j||_{\infty,\mathbb{Z}_p^*} < p^{-rm}, \text{ for any } r \geq 1.$$

This means that

$$P_m(x) - F_j(x) = p^{rm} \sum_{j=r+1}^{N_m} a_m(x-a)^j, \text{ for any } x \in D(a,p^n),$$

with $a_m \in \mathbb{Z}_p$ and N_m the degree of P_m. On the other hand, since $P_m - F_j$ is a locally polynomial function on $\mathbb{Z}_p^*$ with coefficients in $\mathbb{Z}_p$, if $b \neq a$, there exists $Q^a_{b,m} \in \mathbb{Q}_p[X]$ such that the restriction of $P_m - F_j$ to $D(b,p^n)$ equals $p^{rm}Q^a_{b,m}$, for any $x \in D(b,p^n)$. Thus $Q^a_{b,m} \in \mathbb{Z}_p[X]$. Now

$$\int_{D(a,p^n)} P_m d(x)\mu_\alpha(x) - \int_{D(a,p^n)} F_j(x)d\mu_\alpha(x) =$$

$$p^{rm} \sum_{k=r+1}^{N_m} a_m \int_{D(a,p^n)} (x-a)^k d\mu_\alpha(x).$$

But $\int_{D(a,p^n)}(x-a)^k d\mu_\alpha(x) = \alpha^{-n-2}p^{nk}\omega_{k,m}$ with $\omega_{k,m} \in \mathcal{O}_p \otimes \Sigma_f$. Since Σ_f is an $\mathcal{O}_f$-lattice, all the $\omega_{k,m}$ are bounded. Thus, the right hand side tends to 0 p-adically when r increases. If $b \neq a$, $\int_{D(b,p^n)} F_j(x)d\mu_\alpha(x) = 0$ and

$$\int_{D(b,p^n)} P_m(x)d\mu_\alpha(x) - \int_{D(b,p^n)} F_j(x)d\mu_\alpha(x) = p^{rm}\int_{D(b,p^n)} Q^a_{b,m}(x)d\mu_\alpha(x).$$

Since n is fixed and the Taylor expansion of $Q^a_{b,m}$ around b has coefficients in $\mathbb{Z}_p$, all the integrals multiplying the p^{rm} term are bounded; thus the norm of the right hand side also tends to 0 when r increases. Therefore,

$$\int_{\mathbb{Z}_p^*} P_m(x)d\mu_\alpha(x) = 0 = \int_{\mathbb{Z}_p^*} F_j(x)d\mu_\alpha(x) + p^{rm}K_m,$$

with $|K_m|$ bounded, so that $\int_{\mathbb{Z}_p^*} F_j(x)d\mu_\alpha(x) = 0$. □

The main result of this section is the following

Theorem 2.7. *Let $p > 2$ and let $f \in S_k(\Gamma_0(N))$ be a normalized newform. Then for any admissible root α of the Hecke polynomial of f at p, the p-adic function $L_p(f, \alpha)$ does not vanish identically over $\mathbb{Z}_p$. Furthermore, if $k = 2$, it does not vanish identically over $\mathbb{Z}_p^*$.*

Proof. Let us prove first that $L_p(f, \alpha)$ does not vanish identically over $\mathbb{Z}_p$. If this were not the case, we would have

$$\int_{\mathbb{Z}_p^*} \langle x\rangle^r d\mu_\alpha(x) = 0, \quad \text{for all } r \geq 0.$$

Thus $\int_{\mathbb{Z}_p^*} \langle x\rangle^{m(p-1)}d\mu_\alpha(x) = \int_{\mathbb{Z}_p^*} x^{m(p-1)}d\mu_\alpha(x) = 0$, for any $m \geq 0$. Now, we can apply Lemma 2.6 to conclude that, in particular,

$$\int_{D(a,p^n)} x^{\frac{k}{2}-1}d\mu_\alpha(x) = 0.$$

But this would imply that for any $n \geq 1$, a coprime to p, and $1 \leq a \leq p^n-1$,

$$\int_0^\infty f\left(\frac{a}{p^n} + it\right) t^{\frac{k}{2}-1}dt - \frac{1}{\alpha}\int_0^\infty f\left(\frac{a}{p^{n-1}} + it\right) t^{\frac{k}{2}-1}dt = 0. \tag{2.2}$$

Since f is a cusp form, setting $q = e^{2\pi iz}$, $\mathrm{Im}(z) > 0$, there exist constants $M, C_f > 0$ such that

$$\left|\frac{f(q)}{q}\right| \leq C_f, \text{ if } |q| < M.$$

Thus if $y >> 1$, $|f(x+iy)| < C_f e^{-2\pi y}$, and for any a coprime to p,

$$\lim_{m\to\infty} \int_0^\infty f\left(\frac{a}{p^m} + it\right) t^{\frac{k}{2}-1}dt = \int_0^\infty f(it)t^{\frac{k}{2}-1}dt. \tag{2.3}$$

If $\alpha \neq 1$, by taking limits in 2.2, and taking into account 2.3 and the fact that f is a periodic function of period 1, we would have

$$\int_0^\infty f(a+it)t^{\frac{k}{2}-1}dt = 0, \quad \text{for any } a \in \mathbb{Z}.$$

Thus, applying 2.2 recursively, we would obtain

$$\int_0^\infty f\left(\frac{a}{p^m}+it\right)t^{\frac{k}{2}-1}dt = 0,$$

which contradicts Corollary 1.3.

If $\alpha = 1$, setting $n = 1$ in 2.2, we would have

$$\int_0^\infty f\left(\frac{a}{p}+it\right)t^{\frac{k}{2}-1}dt = \int_0^\infty f(it)t^{\frac{k}{2}-1}dt,$$

and, replacing it recursively in 2.2, we would obtain

$$\int_0^\infty f\left(\frac{a}{p^m}+it\right)t^{\frac{k}{2}-1}dt = \int_0^\infty f(it)t^{\frac{k}{2}-1}dt,$$

for any a coprime to p. But this would yield

$$\Lambda\left(f,\chi,\frac{k}{2}\right) = N^{\frac{s}{2}}\frac{\tau(\chi)}{p^n}\sum_{a=1}^{p^n-1}\overline{\chi(a)}\int_0^\infty f(it)t^{\frac{k}{2}-1} = 0,$$

for any Dirichlet character χ, which contradicts Theorem 1.2.

If, moreover, $k = 2$, the result follows from the functional equation in Corollary 1.9, because, if $|s| < 1$, then $2 - s$ is p-adic unit. $\square$

When $k = 2$, it is particularly interesting to restrict the domain of $L_p(f,\alpha;\chi)$ to $\mathbb{Z}_p$, since it yields a natural way of associating p-adic L-functions to modular abelian varieties.

2.3. **The non-vanishing mod p.** We say that a modular integral $\lambda \in \Sigma_f$ is divisible by p if its coordinates are valued in $D(0,p)^r$, r being the rank of Σ_f.

Proposition 2.8. *Let $p > 2$ be an ordinary prime for $f \in S_k(\Gamma_0(N))$, and assume that p is a primitive root mod N. Suppose that $\alpha \not\equiv 1 \pmod p$. If $L_p(f,\alpha)$ vanishes identically mod p over $\mathbb{Z}_p^*$, then all the modular integrals are divisible by p.*

Proof. Let us first observe that if all the modular integrals $\lambda(f, \frac{a}{p^n}, j)$ are divisible by p, then $\lambda(f, 0, j)$ is also divisible by p, since $\lim_{n\to\infty} \lambda(f, \frac{a}{p^n}, j) = \lambda(f, 0, j)$ and Σ_f is closed in $\mathbb{C}^r$. Thus, we are reduced to proving that, under the hypothesis of the proposition, the modular integrals $\lambda(f, \frac{a}{p^m}, j)$, with a coprime to p and $0 \leq j \leq k-2$ are divisible by p. Assuming that this were not the case, there would exist j_0, a minimal $n_0 \geq 1$, and an integer a_0 coprime to p such that $p \nmid \lambda(f, \frac{a_0}{p^{n_0}}, j_0)$. Hence, the measure $\mu_{p,\alpha,j_0}(D(a_0, p^{n_0}))$ would not be divisible by p. On the other hand, by Corollary 2.4, there would exist a sequence $\{P_k(X)\}$ of polynomials in $\mathbb{Z}_p[X^{p-1}]$, converging to the locally polynomial function $x^{j_0}\chi_{D(a_0,p^{n_0})}(x)$ such that

$$\lim_{k\to\infty} \int_{\mathbb{Z}_p^*} P_k(x) d\mu_\alpha(x) = \mu_{p,\alpha,j_0}(D(a_0, p^{n_0})).$$

Since, for any k, $\int_{\mathbb{Z}_p^*} P_k(x) d\mu_\alpha(x) \in D(0,p)^r$, and this is a closed subset, we would have $\mu_{p,\alpha,j_0}(D(a_0, p^{n_0})) \in D(0,p)^r$, which is a contradiction. To complete the proof, let us notice that, p being a primitive root mod N, any modular symbol is a linear combination with integer coefficients of the modular symbols $\lambda\left(f, \frac{a_{k_i}}{p^{n_{k_i}}}, j_i\right)$.

□

3. Results on the order of the p-adic L-function

We are going to exploit the fact that the p-adic L-function $L_p(f, \alpha)$ is not identically zero on $\mathbb{Z}_p$ to prove our main result.

Theorem 3.1. *Let $f \in S_k(\Gamma_0(N))$ be a normalized newform, $p \geq 5$, and α an admissible root of the Hecke polynomial for f at p. Then, for any $s_0 \in \mathbb{Z}_p$,*

$$\mathrm{ord}_{s=s_0} L_p(f, \alpha)(s) < \infty.$$

We recall the following standard notation

$$l_\infty(\mathbb{C}_p) = \{(x_n)_{n\geq 1} \in \mathbb{C}_p^{\mathbb{N}} : \sup_{n\geq 1} |x_n| < \infty\},$$

$$c_0(\mathbb{C}_p) = \{(x_n)_{n\geq 1} \in \mathbb{C}_p^{\mathbb{N}} : \lim_{n\to\infty} |x_n| = 0\}.$$

Both linear spaces are complete with respect to the norm

$$||(x_n)_{n\geq 1}||_\infty = \sup_{n\geq 1} |x_n|.$$

Let $D_i = D(i,p)$, $1 \leq i \leq p-1$. Given $x \in D_i$ and by setting ω_i for the unique $(p-1)$-th root of unity congruent to $i \bmod p$, we may write $x = \omega_i \langle x \rangle$, with $\langle x \rangle = 1 + p\tilde{x}$. Then

$$\tilde{x} = \frac{1}{p}\left(\frac{x}{\omega_i} - 1\right) = \frac{1}{p}\left(\frac{1}{\omega_i}(x-i) + \frac{i}{\omega_i} - 1\right).$$

We set $a_{1,i} = p^{-1}\omega_i^{-1} \in p^{-1}\mathbb{Z}_p$ and $a_{0,i} = p^{-1}\left(i\omega_i^{-1} - 1\right) \in \mathbb{Z}_p$.

For an admissible root α, let us write

$$L_p(f,\alpha)(s) = \int_{\mathbb{Z}_p^*} (1+p\tilde{x})^{s-1}\, d\mu_\alpha(x) = \int_{\mathbb{Z}_p^*} \sum_{k=0}^{\infty} \binom{s-1}{k} p^k \tilde{x}^k d\mu_\alpha(x).$$

Since $n! = p^{\frac{n-\sigma_n}{p-1}} u_n$, with $u_n \in \mathbb{Z}_p^*$ and σ_n being equal to the sum of the p-adic digits of n, we shall have

$$\binom{s-1}{k} p^k = p^{\frac{(p-2)k+\sigma_k}{p-1}} u_k q_k(s),$$

where $q_k(s) = (s-1)(s-2)...(s-k) \in \mathbb{Z}_p[s]$ is a polynomial of degree k.

Proposition 3.2. *If $p \geq 5$, for any sequence $(u_k)_{k\geq 1}$ of p-adic units, the sequence*

$$\left(p^{\frac{(p-2)k+\sigma_k}{p-1}} u_k \int_{\mathbb{Z}_p^*} \tilde{x}^k d\mu_\alpha(x)\right)_{k\geq 1} \quad \textit{is in } c_0(\mathbb{C}_p).$$

Proof. Let $g^k(x) = \tilde{x}^k$, $k \geq 1$. With the former notations, in each disc D_i we shall have

$$g|_{D_i}(x) = a_{1,i}(x-i) + a_{0,i},$$

with $a_{1,i} \in p^{-1}\mathbb{Z}_p$ and $a_{0,i} \in \mathbb{Z}_p$. Thus,

$$g_k|_{D(i)}(x) = \sum_{n=0}^{k} \binom{k}{n} a_{1,i}^n a_{0,i}^{k-n} (x-i)^n.$$

Furthermore, we may write

$$\int_{D(i,p)} (x-i)^n d\mu_\alpha(x) = \alpha^{-2} p^n \gamma_{i,n}, \text{ with } \gamma_{n,i} \in \Sigma_f \otimes \mathcal{O}_p.$$

Thus,

$$\left|\int_{D(i,p)} g_k(x)d\mu_\alpha(x)\right| \leq R\left|\alpha^{-2}\right| \max_{0\leq n\leq k}\left|\binom{k}{n}\right| \leq R\left|\alpha^{-2}\right| p^{\frac{k-\sigma_k}{p-1}}.$$

Combining this with the strong triangle inequality, we obtain

$$\left|\int_{\mathbb{Z}_p^*} \tilde{x}^k d\mu_\alpha(x)\right| \leq R\left|\alpha^{-2}\right| p^{\frac{k-\sigma_k}{p-1}},$$

so that

$$|x_k| = p^{\frac{(2-p)k-\sigma_k}{p-1}}\left|\int_{\mathbb{Z}_p^*} \tilde{x}^k d\mu_\alpha(x)\right| \leq R\left|\alpha^{-2}\right| p^{\frac{(3-p)k-\sigma_k}{p-1}}.$$

□

Using Proposition 3.2 together with the fact that $\frac{p^k}{k!} = p^{\frac{(p-2)k+\sigma_k}{p-1}}u_k$, with $u_k \in \mathbb{Z}_p^*$, we can ensure that the series

$$\langle x\rangle^{s-1} = \sum_{k=0}^{\infty}\binom{s-1}{k}p^k\tilde{x}^k,$$

converges uniformly in $\mathbb{Z}_p^*$. Thus,

$$L_p(f,\alpha)(s) = \sum_{k=0}^{\infty}\int_{\mathbb{Z}_p^*}\binom{s-1}{k}p^k\tilde{x}^k d\mu_\alpha(x). \tag{3.1}$$

The following result will allow us to differentiate the series $L_p(f,\alpha)(s)$ term by term.

Proposition 3.3. *Let* $f_n(s) = n!\binom{s-1}{n}$. *If* $f(s) = \sum_{n=0}^{\infty}\lambda_n f_n(s)$ *with* $(\lambda_n)_{n\geq 0} \in c_0(\mathbb{C}_p)$, *then the series converges uniformly in* $\mathbb{Z}_p$ *and*

$$f^{(j)}(s) = \sum_{n=0}^{\infty}\lambda_n f_n^{(j)}(s), \quad \textit{for } j \geq 0.$$

Proof. Let us first observe that, since $(\lambda_n)_n \in c_0(\mathbb{C}_p)$, the sum $\displaystyle\sum_{n=0}^{\infty}\lambda_n f^{(j)}(s)$ converges for any $s \in \mathbb{Z}_p^*$. We have to prove that, for any $j \geq 0$,

$$\lim_{m\to\infty}\left|\frac{\sum_{n=0}^{\infty}\lambda_n\left(f_n^{(j)}(s+p^m) - f_n^{(j)}(s) - p^m f_n^{(j+1)}(s)\right)}{p^m}\right| = 0, \text{ for any } s \in \mathbb{Z}_p^*. \tag{3.2}$$

We remark that the mean-value theorem does not hold in general in the p-adic setting (cf. [7]). We now proceed by induction on j.

a) For $j = 1$, Let us denote by $e_k(a_1, ..., a_n)$ the k-th symmetrical elementary polynomial in the roots $\{a_1, ..., a_n\}$. For $n, m \geq 1$ and $x \in \mathbb{Z}_p^*$, we have that

$$f_n(s) = e_n(s-1, ..., s-n),$$
$$f_n(s+p^m) = \sum_{j=0}^{n} p^{jm} e_{n-j}(s-1, ..., s-n),$$

and $f_n'(s) = e_{n-1}(s-1, s-2, ..., s-n)$. Thus,

$$f_n(s+p^m) - f_n(s) - p^m f_n'(s) = \sum_{j=2}^{n} p^{jm} e_{n-j}(s-1, ..., s-n)$$

and

$$\left| \frac{f_n(s+p^m) - f_n(s) - p^m f_n'(s)}{p^m} \right| \leq p^{-m}.$$

Since $(\lambda_n)_n$ is bounded, the result follows in this case.

b) Assume that the result holds for $j \geq 1$. A straightforward computation shows that

$$f_n^{(j)}(s) = j! e_{n-j}(s-1, s-2, ..., s-n).$$

Thus,

$$f_n^{(j)}(s+p^m) = j! \sum_{k=0}^{n-j} c_k p^{km} e_{n-j-k}(s-1, s-2, ..., s-n),$$

for suitable integers $c_k \geq 1$.

Lemma 3.4. $c_1 = j+1$. □

Let us note that

$$f_n^{(j+1)}(s) = (j+1)! e_{n-j-1}(s-1, s-2, ..., s-n).$$

Hence, using Lemma 3.4, we have

$$f_n^{(j)}(s+p^m) - f_n^{(j)}(s) - p^m f_n^{(j+1)}(s) = j! \sum_{k=2}^{n-j} c_k p^{km} e_{n-j-k}(s-1, s-2, ..., s-n)$$

and the result holds. □

We proceed now with the proof of Theorem 3.1.

Proof. Fix $s_0 \in \mathbb{Z}_p$. Since $L_p(f,\alpha)$ does not vanish identically, there exists an integer $k \geq 0$ such that

$$\int_{\mathbb{Z}_p^*} \tilde{x}^k d\mu_\alpha(x) \neq 0.$$

Let us define the (non-empty) set

$$\Sigma = \{k \geq 0 : \int_{\mathbb{Z}_p^*} \tilde{x}^k d\mu_{f,\alpha} \neq 0\}.$$

If Σ is finite, we consider its maximal integer k and, by taking the k-th derivative of $L_p(f,\alpha)(s)$ at $s = s_0$, we obtain

$$\frac{d^k}{ds^k} L_p(f,\alpha)(s)|_{s=s_0} = p^k \int_{\mathbb{Z}_p^*} \tilde{x}^k d\mu_{f,\alpha} \neq 0.$$

Now we suppose that Σ is an infinite set. The k_n-th term in the series has the form

$$p^{k_n} \binom{s-1}{k_n} \int_{\mathbb{Z}_p^*} \tilde{x}^{k_n} d\mu_{f,\alpha} = p^{\frac{(p-2)k_n + \sigma_{k_n}}{p-1}} u_{k_n} q_{k_n}(s) \int_{\mathbb{Z}_p^*} \tilde{x}^{k_n} d\mu_{f,\alpha}\,,$$

with $q_{k_n}(s) = (s-1)(s-2)...(s-k_n)$. If we denote by $a^{i,j}$ the k_i-th derivative of $q_{k_j}(s)$ at $s = s_0$, we may define the linear endomorphism

$$\psi : c_0(\mathbb{C}_p) \longrightarrow c_0(\mathbb{C}_p), \quad x = (x_n) \mapsto \sum_{r=n}^{\infty} a^{r,n} x_n.$$

Since $a^{i,j} \in \mathbb{Z}_p$, ψ is well defined and $||\psi(x)|| \leq ||x||$, it is a continuous map. We can see this endomorphism as being represented by an infinite matrix $(a^{i,j})_{i,j\geq 1}$ which is upper triangular.

Proposition 3.5. *The endomorphism ψ is injective.*

Proof. For $x = (x_n)_{n\geq 1} \in c_0(\mathbb{C}_p)$, we have $\psi(x) = (D \circ \phi)(x)$ where $D : c_0(\mathbb{C}_p) \longrightarrow c_0(\mathbb{C}_p)$, $x = (x_n) \mapsto (k_n! x_n)$, and $\phi : c_0(\mathbb{C}_p) \longrightarrow c_0(\mathbb{C}_p)$ is given

by left natural multiplication with the infinite matrix

$$M_\phi = \begin{pmatrix} 1 & \frac{a_{1,2}}{k_1!} & \frac{a_{1,3}}{k_1!} & \frac{a_{1,4}}{k_1!} & \dots \\ 0 & 1 & \frac{a_{2,3}}{k_2!} & \frac{a_{2,4}}{k_1!} & \dots \\ 0 & 0 & 1 & \frac{a_{3,4}}{k_3!} & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us check that $\phi(c_0(\mathbb{C}_p)) \subseteq c_0(\mathbb{C}_p)$. Note that $q_{k_j}(s) = s^{k_j} + e_1^{k_j} s^{k_j-1} + ... + e_{k_j}^{k_j}$, with $e_r^{k_j}$ the r-th elementary symmetrical polynomial in the roots $\{1, 2, ..., k_j\}$. Thus,

$$\frac{1}{k_i!}\frac{d^{k_i}}{ds^{k_i}} q_{k_j}(s) = \tbinom{k_j}{k_j-k_i} s^{k_j-k_i} + \tbinom{k_j-1}{k_j-1-k_i} e_1^{k_j} s^{k_j-k_i-1} + ... + \tbinom{k_i}{0} e_{k_i}^{k_j}.$$

Since

$$\binom{k_j - r}{k_j - k_i - r} = p^{\frac{-\sigma_{k_j-r}+\sigma_{k_j-k_i-r}+\sigma_{k_i}}{p-1}} \theta_r,$$

with $\theta_r \in \mathbb{Z}_p^*$, for $1 \le r \le k_j - k_i$, and

$$-\sigma_{k_j-r} + \sigma_{k_j-k_i-r} + \sigma_{k_i} = (p-1)\mathrm{ord}_p \binom{k_j + k_i - r}{k_i}$$

(cf. [4]), it follows that

$$\left| \frac{1}{k_i!}\frac{d^{k_i}}{ds^{k_i}} q_{k_j}(s)|_{s=s_0} \right| \le 1.$$

Hence, ϕ is well defined and its associated matrix M_ϕ is upper triangular with ones in the diagonal. To prove the injectivity, we proceed by steps:

Step 1. We consider the adjoint matrix of the element $a_{i,j}$:

$$M_\phi^{i,j} = \begin{pmatrix} \frac{a_{1,1}}{k_1!} & \dots & \frac{a_{1,j-1}}{k_1!} & \frac{a_{1,j+1}}{k_1!} & \dots \\ 0 & \dots & \frac{a_{2,j-1}}{k_2!} & \frac{a_{2,j+1}}{k_2!} & \dots \\ \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{a_{i-1,j-1}}{k_{i-1}!} & \frac{a_{i-1,j+1}}{k_{i-1}!} & \dots \\ 0 & \dots & \frac{a_{i+1,j-1}}{k_{i+1}!} & \frac{a_{i+1,j+1}}{k_{i+1}!} & \dots \\ \vdots & \dots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From a particular row onwards, this matrix becomes upper triangular, with ones on the diagonal, hence we can formally compute the determinant

$\det(M_\phi^{i,j})$ expanding along the first row. The result is a finite sum of finite products of the terms $\frac{a_{i,j}}{k_i!} \in \mathbb{Z}_p$, therefore, it is a p-adic integer.

Step 2. We put together all the adjoint matrices and take the transpose, obtaining an upper-triangular matrix with entries in $\mathbb{Z}_p$ and ones on the diagonal. We write a few terms of this matrix:

$$M_\phi^{'} = \begin{pmatrix} 1 & -\frac{a_{1,2}}{k_1!} & -\frac{a_{1,3}}{k_1!} + \frac{a_{1,2}a_{2,3}}{k_1!k_2!} & -\frac{a_{1,2}a_{2,3}a_{3,4}}{k_1!k_2!k_3!} + \frac{a_{1,3}a_{3,4}}{k_1!k_3!} + \frac{a_{1,2}a_{3,4}}{k_1!k_3!} - \frac{a_{1,4}}{k_1!} & \dots \\ 0 & 1 & -\frac{a_{2,3}}{k_2!} & \frac{a_{2,4}a_{3,4}}{k_2!k_3!} - \frac{a_{2,4}}{k_2!} & \dots \\ 0 & 0 & 1 & -\frac{a_{3,4}}{k_3!} & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Now, $M_\phi^{'}$ defines again a continuous linear endomorphism of $c_0(\mathbb{C}_p)$, that we call ϕ'.

Step 3. Since $M_\phi, M_\phi^{'}$ are upper triangular infinite matrices with ones on the diagonal and entries in $\mathbb{Z}_p$, we have that $(\phi' \circ \phi)(x) = x$, for all $x \in c_0(\mathbb{C}_p)$; i. e., $\phi^{'}$ is a right inverse of ϕ.

Since D is trivially injective, so is ψ, completing the proof of Proposition 3.5 and Theorem 3.1. □

To any elliptic curve $E/\mathbb{Q}$ of conductor N, we can attach its complex analytic L-series $L(E, s)$. By modularity, there exists a weight 2 normalized newform f for $\Gamma_0(N)$ such that $L(E, s) = L(f_E, s)$. For an admissible root α of the Hecke polynomial of f_E at p, the corresponding p-adic L-function of E is defined by setting

$$L_p(E, \alpha)(s) = L_p(f_E, \alpha)(s), \quad s \in \mathbb{Z}_p.$$

Corollary 3.6. *If $p \geq 5$ and α is an admissible root of the Hecke polynomial of f_E at p, then*

$$\mathrm{ord}_{s=1} L_p(E, \alpha)(s) < \infty.$$

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